How to Show Two Sets are Equal

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Background

Suppose we aim to show $\mathcal{A} = \mathcal{B}$.

This is true if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.

Thus, it suffices to verify the latter statements.

Consider an arbitrary element $a \in \mathcal{A}$.

If we can show $a \in \mathcal{B}$, then it follows that each element in $\mathcal{A}$ is also in $\mathcal{B}$, *i.e.* $\mathcal{A} \subseteq \mathcal{B}$.

Analogous argument works to show $\mathcal{B} \subseteq \mathcal{A}$.

We next show a template for applying this idea.

Proof Template

To verify $\mathcal{A} = \mathcal{B}$, we proceed by showing $\mathcal{A} \subseteq \mathcal{B}$ (Step 1) and $\mathcal{B} \subseteq \mathcal{A}$ (Step 2).

Step 1. Let $a \in \mathcal{A}$ be given. To prove $\mathcal{A} \subseteq \mathcal{B}$, it suffices to show $a \in \mathcal{B}$. [Insert argument that $a \in \mathcal{B}$]

Step 2. Let $b \in \mathcal{B}$ be given. To prove $\mathcal{B} \subseteq \mathcal{A}$, it suffices to show $b \in \mathcal{A}$. [Insert argument that $b \in \mathcal{A}$] ■

Example on Real Line

Let $\mathcal{A} = \{x : x \geq 0\}$ and $\mathcal{B} = \{y : y = x^2 \text{ for } x \in \mathbb{R}\}$.

Prove $\mathcal{A} = \mathcal{B}$.

Proof: To verify $\mathcal{A} = \mathcal{B}$, we proceed by showing $\mathcal{A} \subseteq \mathcal{B}$ (Step 1) and $\mathcal{B} \subseteq \mathcal{A}$ (Step 2).

Step 1. Let $a \in \mathcal{A}$ be given. To prove $\mathcal{A} \subseteq \mathcal{B}$, it suffices to show $a \in \mathcal{B}$. Indeed, since $a \geq 0$, it has a real root. Thus, $a = (\sqrt{a})^2$, and so $a \in \mathcal{B}$.

Step 2. Let $b \in \mathcal{B}$ be given. To prove $\mathcal{B} \subseteq \mathcal{A}$, it suffices to show $b \in \mathcal{A}$. By definition of $\mathcal{B}$, there is $x \in \mathbb{R}$ such that $b = x^2 \geq 0$ (since squares are nonnegative). This implies $b \geq 0$, and so $b \in \mathcal{A}$. ∎

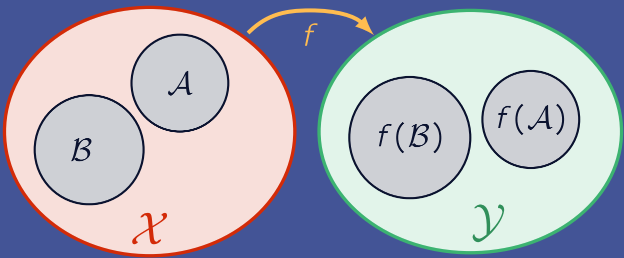
Example with Image of Function

Consider a function $f\colon \mathcal{X} \to \mathcal{Y}$ and sets $A, B \subseteq \mathcal{X}$.

Prove

$$f(A \cup B) = f(A) \cup f(B).$$

First, we illustrate the problem with a diagram.

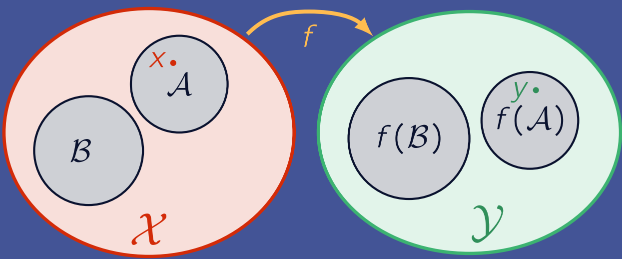


Next we follow the steps in the given template.

Proof (Part 1 of 2)

We proceed by showing $f(\mathcal{A}) \cup f(\mathcal{B}) \subseteq f(\mathcal{A} \cup \mathcal{B})$ (Step 1) and $f(\mathcal{A} \cup \mathcal{B}) \subseteq f(\mathcal{A}) \cup f(\mathcal{B})$ (Step 2).

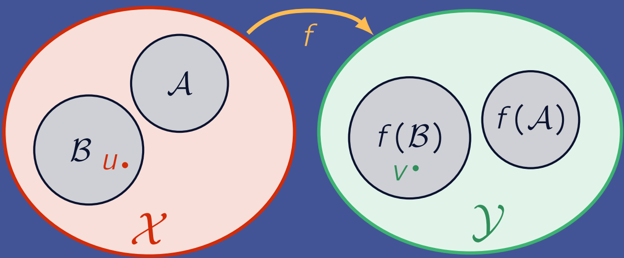
Step 1. Let $y \in f(A) \cup f(B)$ be given. To prove $f(\mathcal{A}) \cup f(\mathcal{B}) \subseteq f(\mathcal{A} \cup \mathcal{B})$, it suffices to show $y \in f(A) \cup f(B)$. Note either $y \in f(A)$ or $y \in f(B)$. So, there is either $x \in A$ such that $f(x) = y$ or $x \in B$ such that $f(x) = y$. Thus, there is $x \in A \cup B$ such that $f(x) = y$, and so $y \in f(A \cup B)$.



Proof (Part 2 of 2)

Step 2. Let $v \in f(A \cup B)$ be given. To prove $f(\mathcal{A} \cup \mathcal{B}) \subseteq f(\mathcal{A}) \cup f(\mathcal{B})$, it suffices to show $v \in f(A) \cup f(B)$. Note there is $u \in A \cup B$ such that $f(u) = v$. This implies either there is $u \in A$ such that $f(u) = v$ or there is $u \in B$ such that $f(u) = v$. Thus, either $v \in f(A)$ or $v \in f(B)$, and so $v \in f(A) \cup f(B)$.

■



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