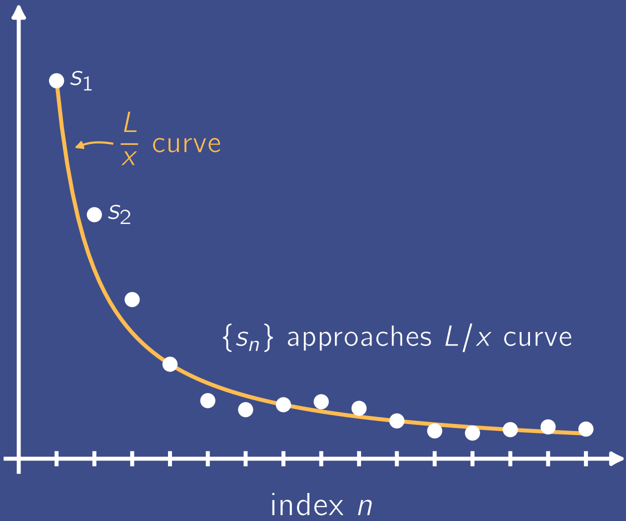
Series Divergence via Asymptotic Behavior

Prove $\lim_{n \to \infty} n \cdot s_n = L > 0$ implies $\sum_{n=1}^{\infty} s_n = \infty$



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Scratch Work

Since

$$\lim_{n\to\infty} n \cdot s_n = L,$$

when n is big we have

$$s_n \cdot n \approx L \qquad \Longrightarrow \qquad s_n \approx \frac{L}{n}.$$

So, the asymptotic behavior of the series

$$\sum_{n=1}^{\infty} s_n$$

reflects that of L times the harmonic series, which diverges. We can make a formal argument by showing s_n is at least some multiple of L/n when n is beyond a threshold and, thus, the tail of the series is bounded below by a multiple of the tail of the harmonic series (which sums to ∞).

Proof

Since $\{n \cdot s_n\}$ converges to L, there is $N \in \mathbb{N}$ such that

$$|n \cdot s_n - L| \leq \frac{L}{2}, \quad \text{for all } n \geq N.$$

This implies, for all $n \geq N$,

$$n \cdot s_n - L \geq -\frac{L}{2} \quad \Longrightarrow \quad n \cdot s_n \geq \frac{L}{2} \quad \Longrightarrow \quad s_n \geq \frac{L}{2n}.$$

Thus,

$$\sum_{n=N}^{\infty} s_n \geq \sum_{n=N}^{\infty} \frac{L}{2n} = \frac{L}{2} \cdot \sum_{n=N}^{\infty} \frac{1}{n} = \infty,$$

where the final equality holds since removing a finite number of terms from a divergent series leaves its asymptotic behavior unchanged, making the truncated harmonic series diverge. Since the sum of the first $N - 1$ summands s_n is finite, we conclude

$$\sum_{n=1}^{\infty} s_n = \sum_{n=1}^{N-1} s_n + \sum_{n=N}^{\infty} s_n = \infty. \quad \blacksquare$$